Ejercicios resueltos: ejemplos
Lectura 3

Ejercicio *nº* **1** *Integrar:*

(a) $f(x,y) = 2xy^2$ *sobre el primer cuadrante de la circunferencia de radio* R.

(b) $f(x,y,z) = (x^2+y^2+z^2)^2$ *a lo largo de la hélice circular* $\mathbf{r}(t) = (\cos t, \operatorname{sen} t, 3t)$, *desde el punto* $(1,0,0)$ *hasta el punto* $(1,0,6\pi)$.

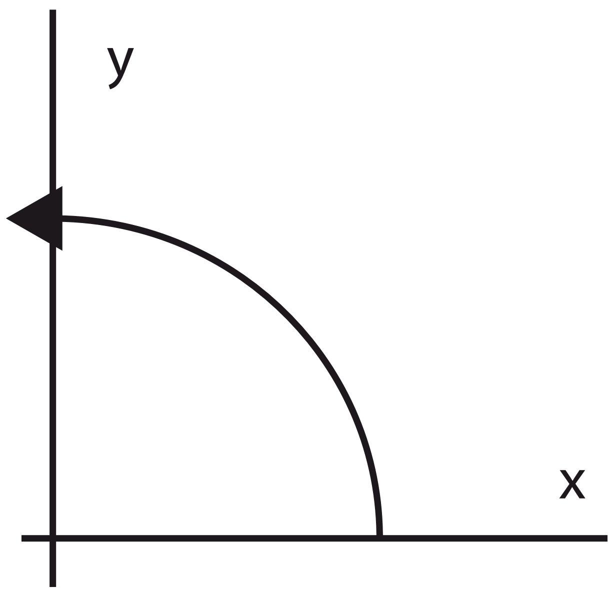

Figura 1: Ejercicio 1, apartado a

Solución:
Ejercicio nº 1, apartado a:
Las ecuaciones paramétricas del arco de circunferencia γ,

$$x = R\cos\theta \qquad y = R\operatorname{sen}\theta \qquad \mathrm{d}s = R\,\mathrm{d}\theta$$

y el campo de variación del parámetro, $0 \le \theta \le \dfrac{\pi}{2}$, luego

$$I = \int_\gamma 2xy^2\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} 2R\cos\theta\, R^2\operatorname{sen}^2\theta\, R\,\mathrm{d}\theta = 2R^4\int_0^{\frac{\pi}{2}}\cos\theta\operatorname{sen}^2\theta\,\mathrm{d}\theta =$$

$$= 2R^4\left.\frac{\operatorname{sen}^3\theta}{3}\right|_0^{\frac{\pi}{2}} = \frac{2R^4}{3}$$

Ejercicio nº 1, apartado b:
La variación del parámetro la obtenemos

$$\begin{array}{rclcrcl}
(\cos t_0, \operatorname{sen} t_0, 3t_0) & = & (1,0,0) & \rightarrow & t_0 & = & 0 \\
(\cos t_1, \operatorname{sen} t_1, 3t_1) & = & (1,0,6\pi) & \rightarrow & t_1 & = & 2\pi
\end{array}$$

El diferencial de arco

$$\mathrm{d}\,s = \sqrt{(-\operatorname{sen} t)^2 + (\cos t)^2 + (3)^2}\;\mathrm{d}\,t = \sqrt{10}\;\mathrm{d}\,t$$

luego

$$\begin{aligned} I = & \int_\gamma \left(x^2+y^2+z^2\right)^2 \mathrm{d}\,s = \int_0^{2\pi} \left(1+9t^2\right)^2\ \sqrt{10}\,\mathrm{d}\,t = \\ = & \sqrt{10}\int_0^{2\pi}\left(1+18t^2+81t^4\right)\ \mathrm{d}\,t = \sqrt{10}\left[t+6t^3+\frac{81t^5}{5}\right]_0^{2\pi} = 2\pi\sqrt{10}\frac{5+120\pi^2+1296\pi^4}{5} \end{aligned}$$

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Ejercicio $n^{\underline{o}}$ **2** *Calcular* $\int_\Gamma \left(5-xy-y^2\right)\mathrm{d}\,x - \left(2xy-x^2\right)\mathrm{d}\,y$, *siendo* Γ *el cuadrado de vértices* $(0,0)$, $(1,0)$, $(1,1)$ *y* $(0,1)$, *directamente y aplicando el teorema de Green.*

Solución:

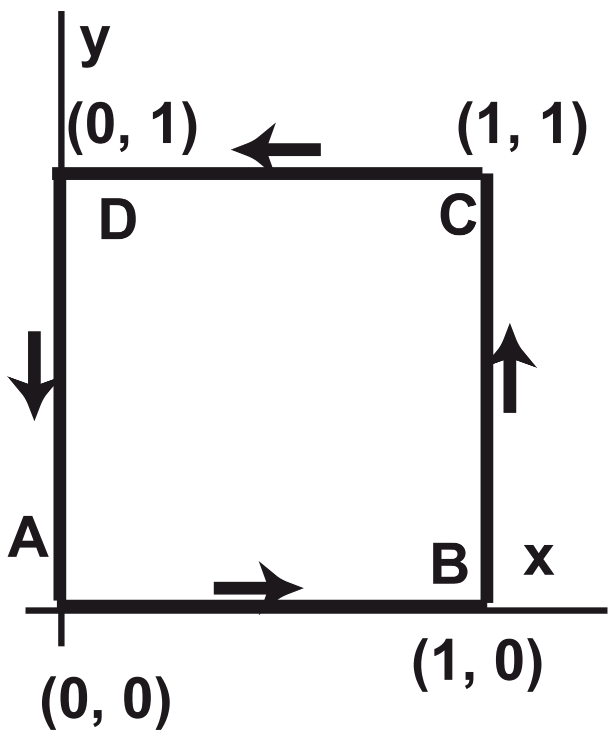

Figura 2: Ejercicio 2

Si $\left(5-xy-y^2\right)\mathrm{d}\,x\ -\ \left(2xy-x^2\right)\mathrm{d}\,y \equiv X\left(x,\,y\right)\mathrm{d}\,x + Y\left(x,\,y\right)\mathrm{d}\,y$, será

$$\begin{aligned} I = \quad & \int_{(0,0)}^{(1,0)} X\left(x,\,y\right)\mathrm{d}\,x + Y\left(x,\,y\right)\mathrm{d}\,y + \int_{(1,0)}^{(1,1)} X\left(x,\,y\right)\mathrm{d}\,x + Y\left(x,\,y\right)\mathrm{d}\,y + \\ + \quad & \int_{(1,1)}^{(0,1)} X\left(x,\,y\right)\mathrm{d}\,x + Y\left(x,\,y\right)\mathrm{d}\,y + \int_{(0,1)}^{(0,0)} X\left(x,\,y\right)\mathrm{d}\,x + Y\left(x,\,y\right)\mathrm{d}\,y = \\ = \quad & \int_0^1 X\left(x,\,0\right)\mathrm{d}\,x + \int_0^1 Y\left(1,\,y\right)\mathrm{d}\,y + \int_1^0 X\left(x,\,1\right)\mathrm{d}\,x + \int_1^0 Y\left(0,\,y\right)\mathrm{d}\,y \end{aligned}$$

hagamos las cuatro integrales

$$
\begin{aligned}
I_1 = &\int_0^1 X\,(x,\,0)\,\mathrm{d}\,x = \int_0^1 5\,\mathrm{d}\,x = 5\\
I_2 = &\int_0^1 Y\,(1,\,y)\,\mathrm{d}\,y = -\int_0^1 (2y-1)\,\mathrm{d}\,y = -\left(y^2-y\right)_0^1 = 0\\
I_4 = &\int_1^0 X\,(x,\,1)\,\mathrm{d}\,x = \int_1^0 (5-x-1)\,\mathrm{d}\,x = \left(4x-\frac{x^2}{2}\right)_1^0 = -4+\frac{1}{2} = -\frac{7}{2}\\
I_4 = &\int_1^0 Y\,(0,\,y)\,\mathrm{d}\,y = -\int_1^0 0\,\mathrm{d}\,y = 0
\end{aligned}
$$

luego

$$
I = I_1 + I_2 + I_3 + I_4 = 5 + 0 - \frac{7}{2} + 0 = \frac{3}{2}
$$

Aplicando el teorema de Green

$$
\begin{aligned}
I = &\oint_C X\,\mathrm{d}\,x + Y\,\mathrm{d}\,y = \iint_D \left[\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right]\mathrm{d}\,x\,\mathrm{d}\,y = \iint_D \left[2x-2y-(-x-2y)\right]\mathrm{d}\,x\,\mathrm{d}\,y =\\
= &\iint_D 3x\,\mathrm{d}\,x\,\mathrm{d}\,y = \int_0^1 3x\,\mathrm{d}\,x \int_0^1 \mathrm{d}\,y = \frac{3}{2}
\end{aligned}
$$

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