Department of Applied Mathematics
and Computational Sciences
University of Cantabria
UC-CAGD Group

COMPUTER-AIDED GEOMETRIC DESIGN AND COMPUTER GRAPHICS:

BEZIER CURVES AND SURFACES

Andrés Iglesias
e-mail: iglesias@unican.es
Web pages: http://personales.unican.es/iglesias
http://etsiso2.macc.unican.es/~cagd

Bézier curves

BEZIER CURVES

Let $P=\{P_0, P_1, \ldots, P_n\}$ *be a set of points* $\mathbf{P}_i \in \mathbb{R}^d$, d=2,3.

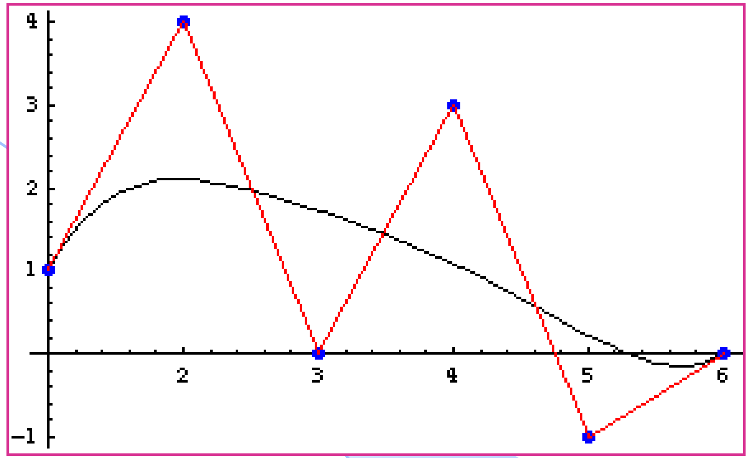
The Bézier curve associated with the set ***P*** *is defined by:*

$$\sum_{i=0}^{n} \mathbf{P}_i B_i^n(t)$$

where $B_i^n(t)$ *represent the Bernstein polynomials, which are given by:*

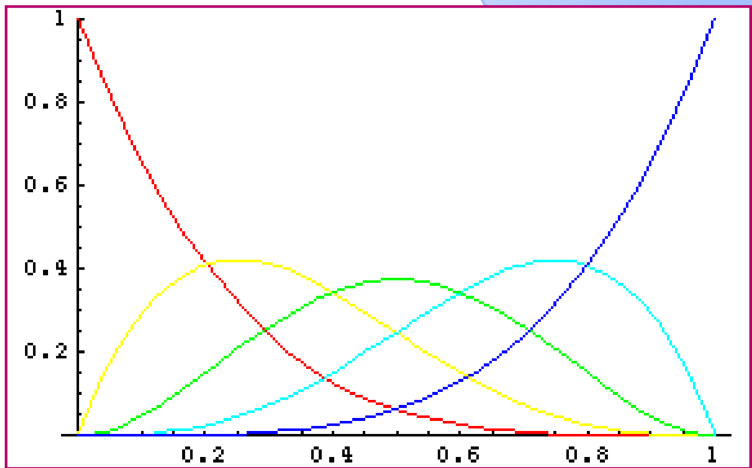
$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i \qquad i = 0, \ldots, n$$

n being the polynomial degree.



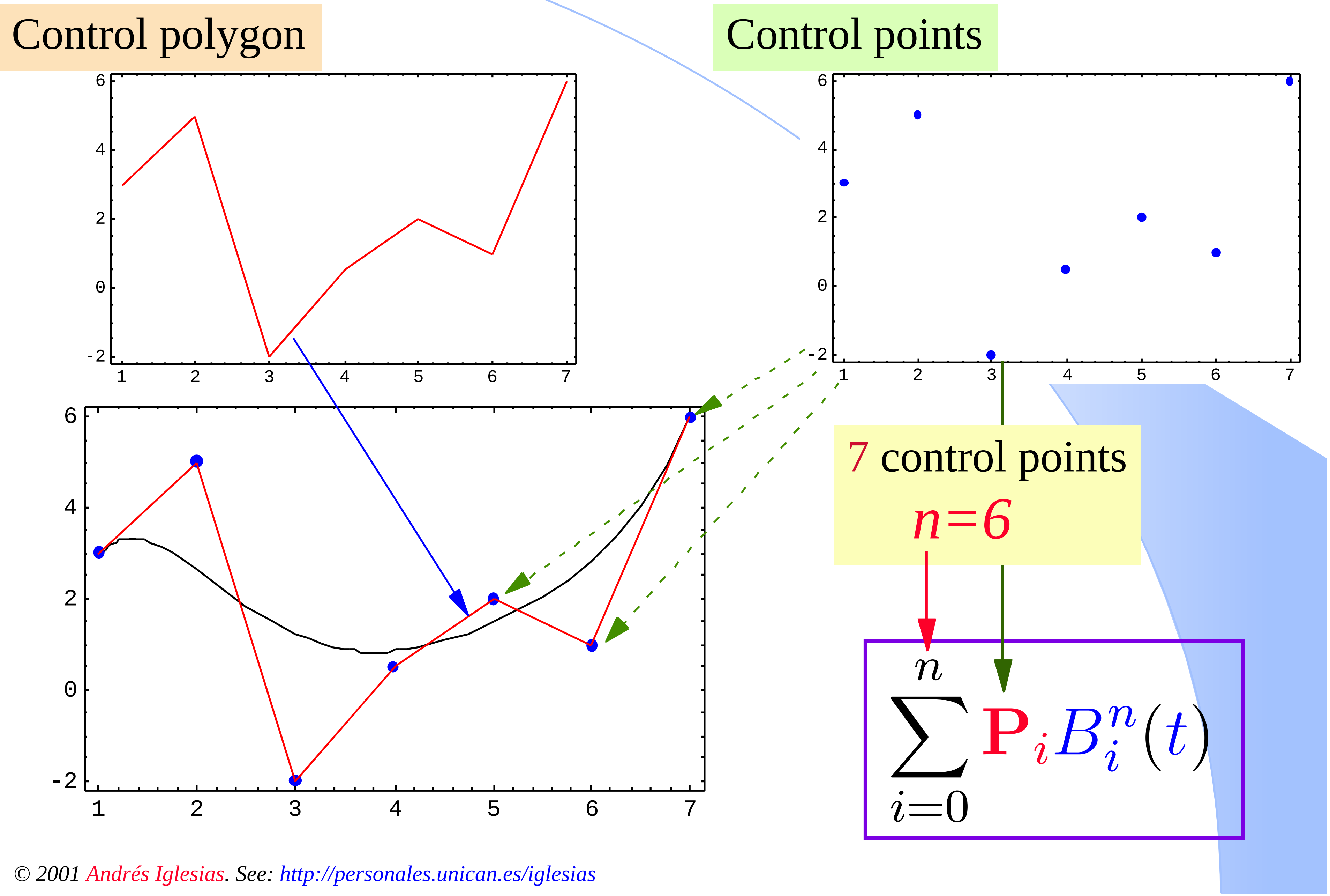
Bézier curve *with n=5*

(six control or Bézier points)



Bernstein polynomials $B_i^4(t)$

Bézier curves



$$\sum_{i=0}^{n} \mathbf{P}_i B_i^n(t)$$

Bernstein polynomials

Given by:

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i \qquad i = 0,\ldots,n$$

n the degree
i the index
t the variable

Properties:

Extreme values:

$B_i^n(0) = B_i^n(1) = 0 \quad i=1,\ldots,n-1$
$B_0^n(0) = B_n^n(1) = 1$
$B_i^n(1) = B_i^n(0) = 0$

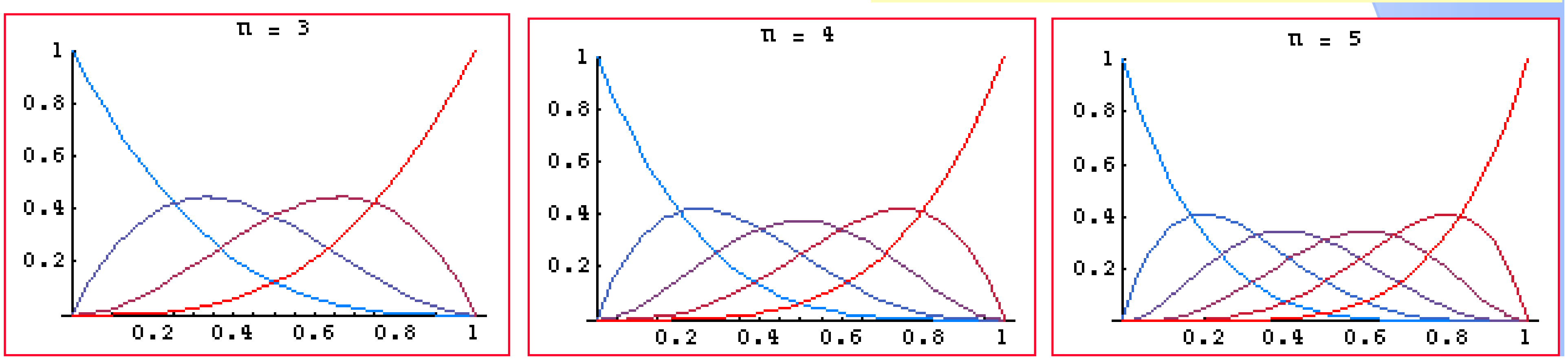
Positivity: $B_i^n(t) \geq 0$ in $[0,1]$

Simmetry: $B_i^n(t) = B_{n-i}^n(1-t)$

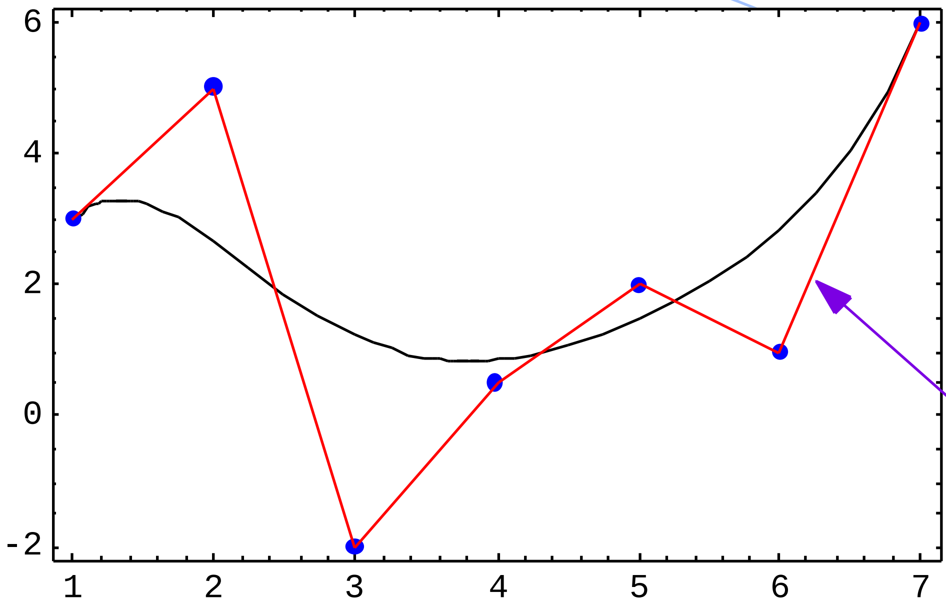
Normalizing property:

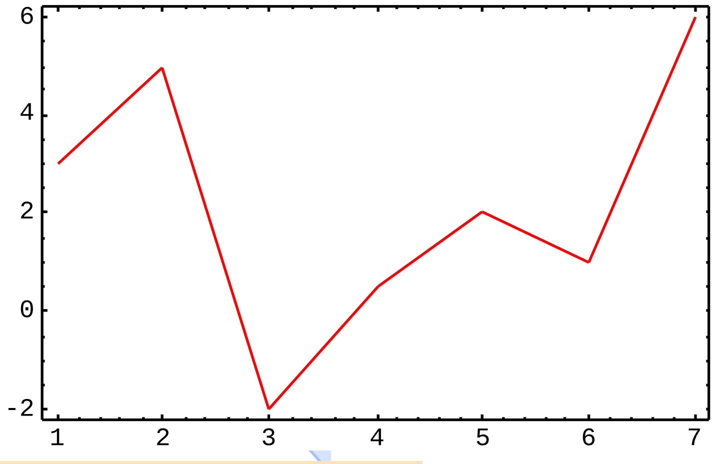
$$\sum_{i=0}^{n} B_i^n(t) = 1$$

Maxima: $B_i^n(t)$ attains exactly one maximum on the interval $[0,1]$, at $t=i/n$.



Properties of the Bézier curves

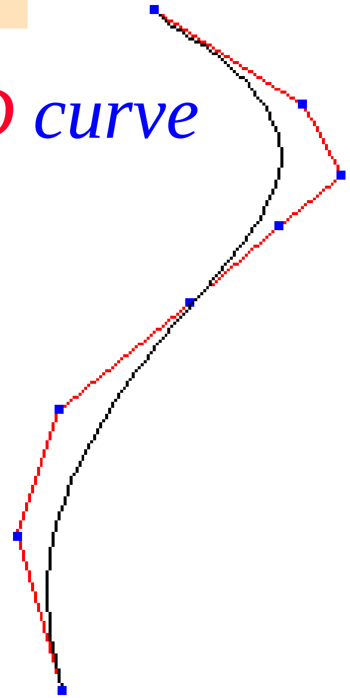




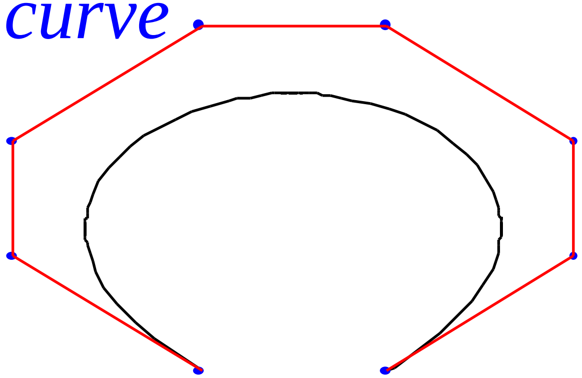
Control polygon

The Bézier curve generally follows the shape of the control polygon, which consists of the segments joining the control points.

3D curve



2D curve



Bézier scheme is useful for design.

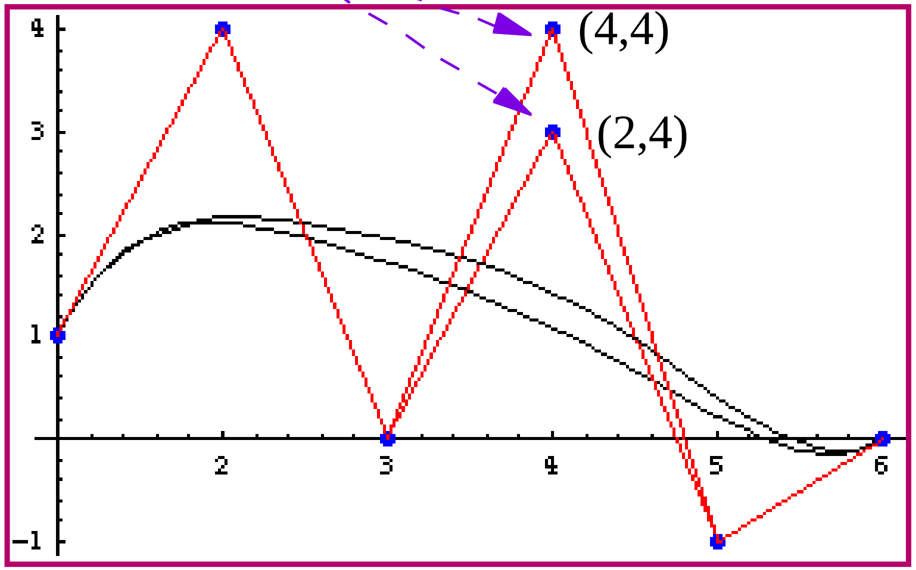
Properties of the Bézier curves

LOCAL vs. GLOBAL CONTROL

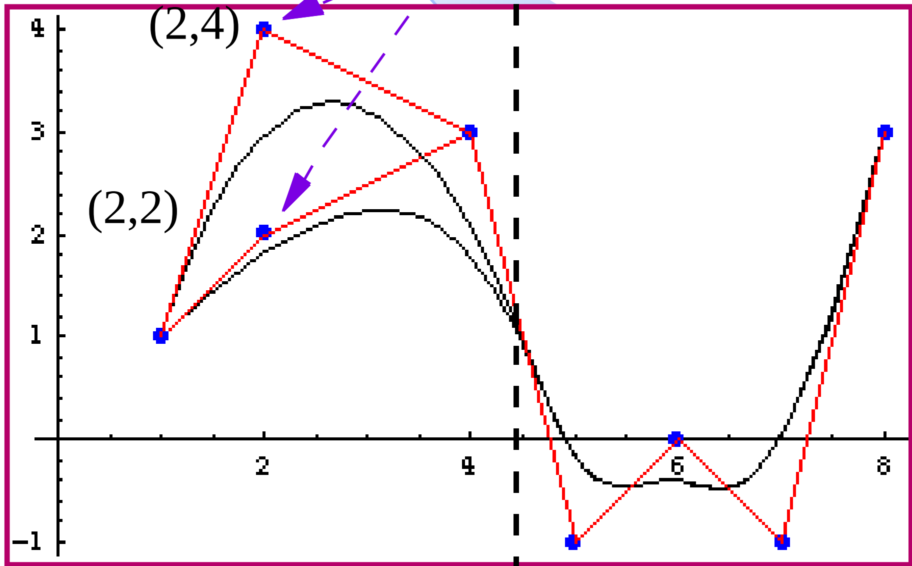
Bézier curves exhibit global control: moving a control point alters the shape of the whole curve.

B-splines allow local control: only a part of the curve is modified when changing a control point.

Control point traslation.



Control point traslation.



The curve changes here

*The curve **does not** change here*

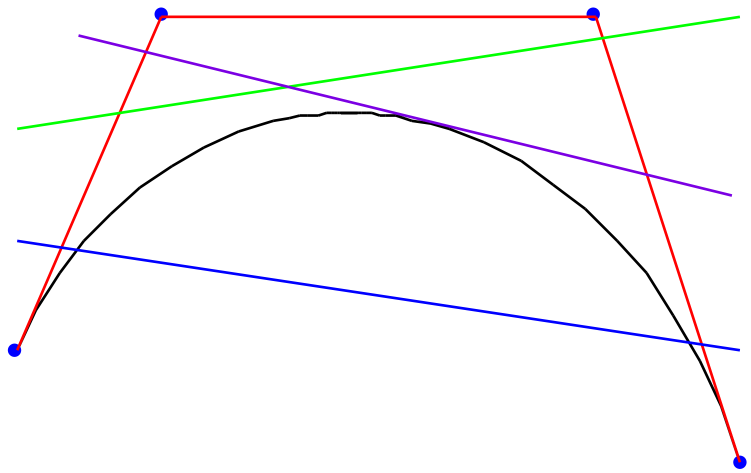
Properties of the Bézier curves

Interpolation. A Bézier curve always interpolates the end control points.

Tangency. The endpoint tangent vectors are parallel to $P_1 - P_0$ *and* $P_n - P_{n-1}$

Convex hull property. The curve is contained in the convex hull of its defining control points.

Variation disminishing property. No straight line intersects a Bézier curve more times than it intersects its control polygon.



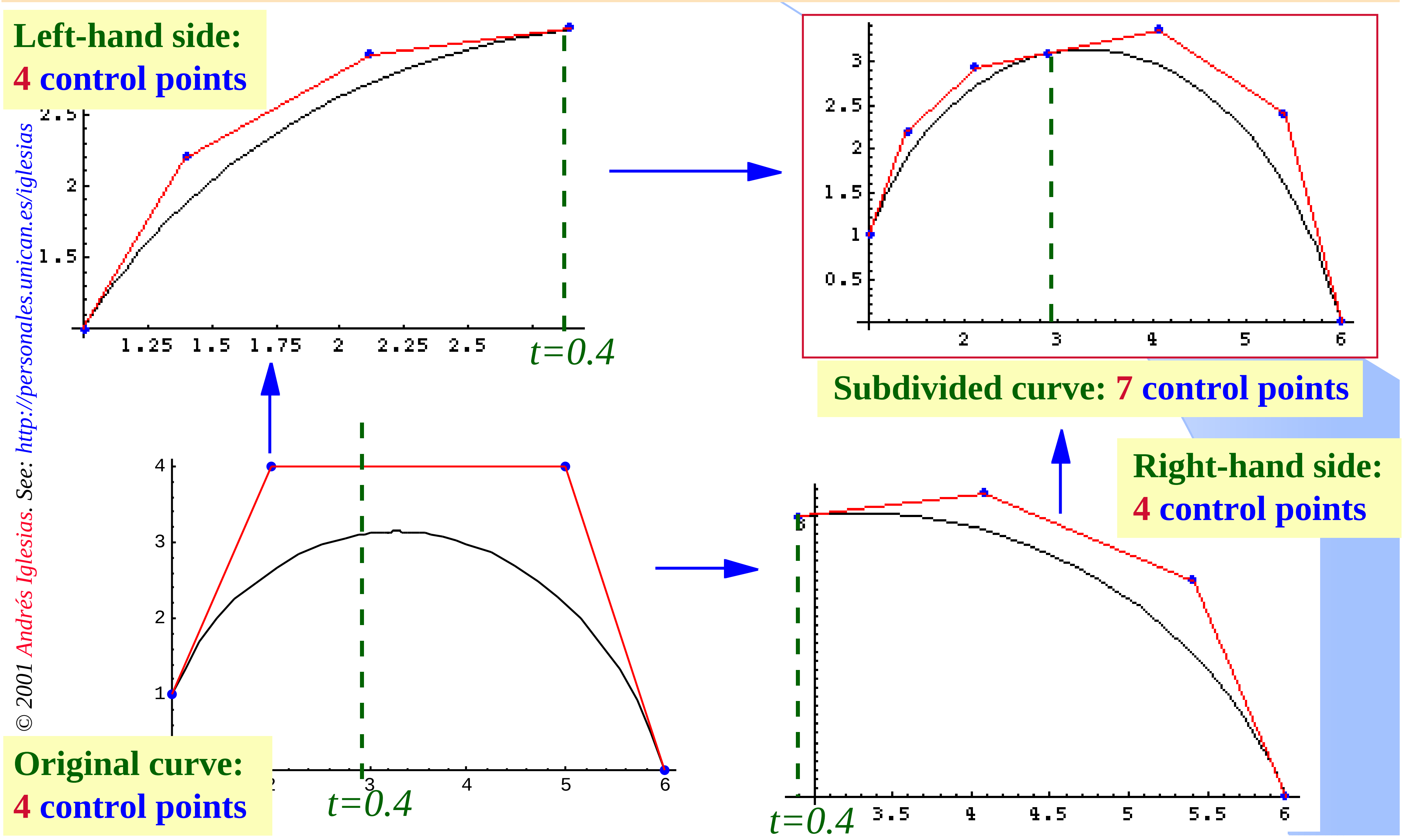
Intersections

Curve: 0	**Polygon: 2**
Curve: 1	**Polygon: 2**
Curve: 2	**Polygon: 2**

For a three-dimensional Bézier curve, replaces the words straight line with the word plane.

Properties of the Bézier curves

A given Bézier curve can be ***subdivided at a point $t=t_0$*** *into two Bézier segments which join together at the point corresponding to the parameter value $t=t_0$.*



Properties of the Bézier curves

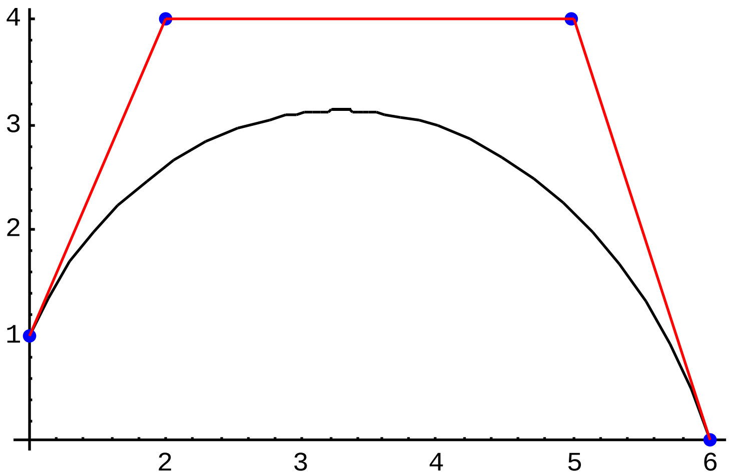
Degree raising: *any Bézier curve of degree n (with control points P_i) can be expressed in terms of a new basis of degree $n+1$. The new control points Q_i are given by:*

$$\boxed{Q_i = \frac{i}{n+1} P_{i-1} + \left(1 - \frac{1}{n+1}\right) P_i}$$

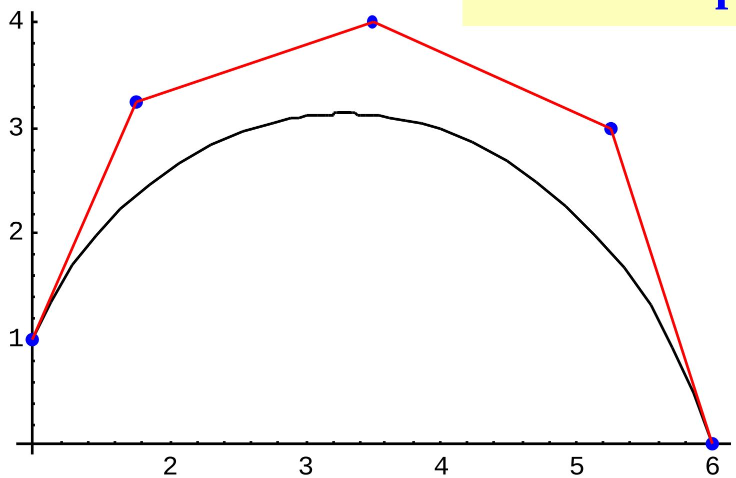
$$i=0,\ldots,n+1$$

$$P_{-1} = P_{n+1} = 0$$

Original cubic curve:
4 control points

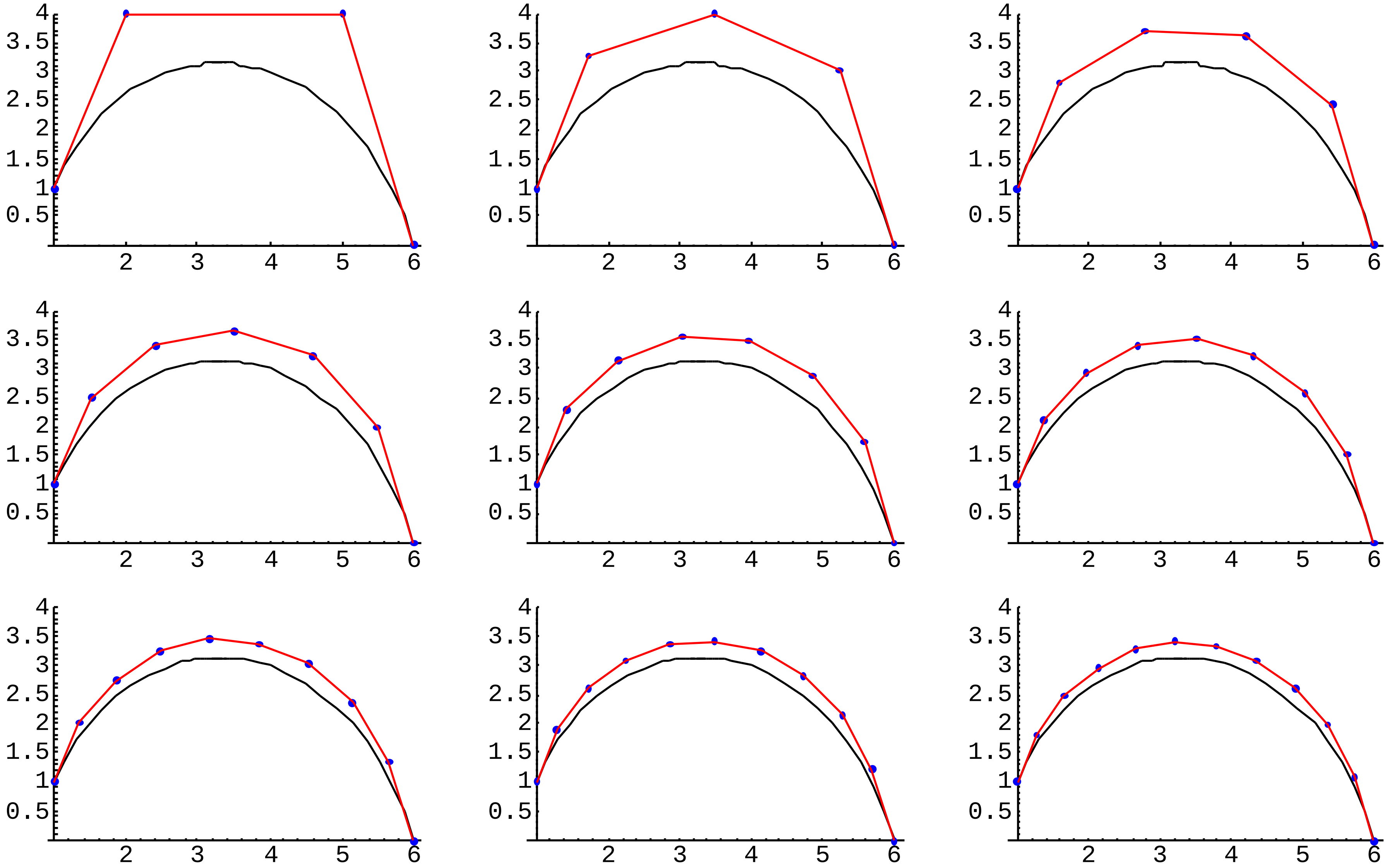


Final quartic curve:
5 control points



Bézier curves

Degree raising of the Bézier curve of degree n=3 to degree n=11



Rational Bézier curves

There are a number of important curves and surfaces which cannot be represented faithfully using polynomials, namely, circles, ellipses, hyperbolas, cylinders, cones, etc.

All the *conics* can be well represented using *rational functions*, which are the ratio of two polynomials.

Rational Bézier curve

$$R(t) = \frac{\sum_{i=0}^{n} P_i \, w_i \, B_i^n(t)}{\sum_{i=0}^{n} w_i \, B_i^n(t)}$$

$w_i \longrightarrow$ *weights*

If all $w_i = 1$, we recover the Bézier curve.

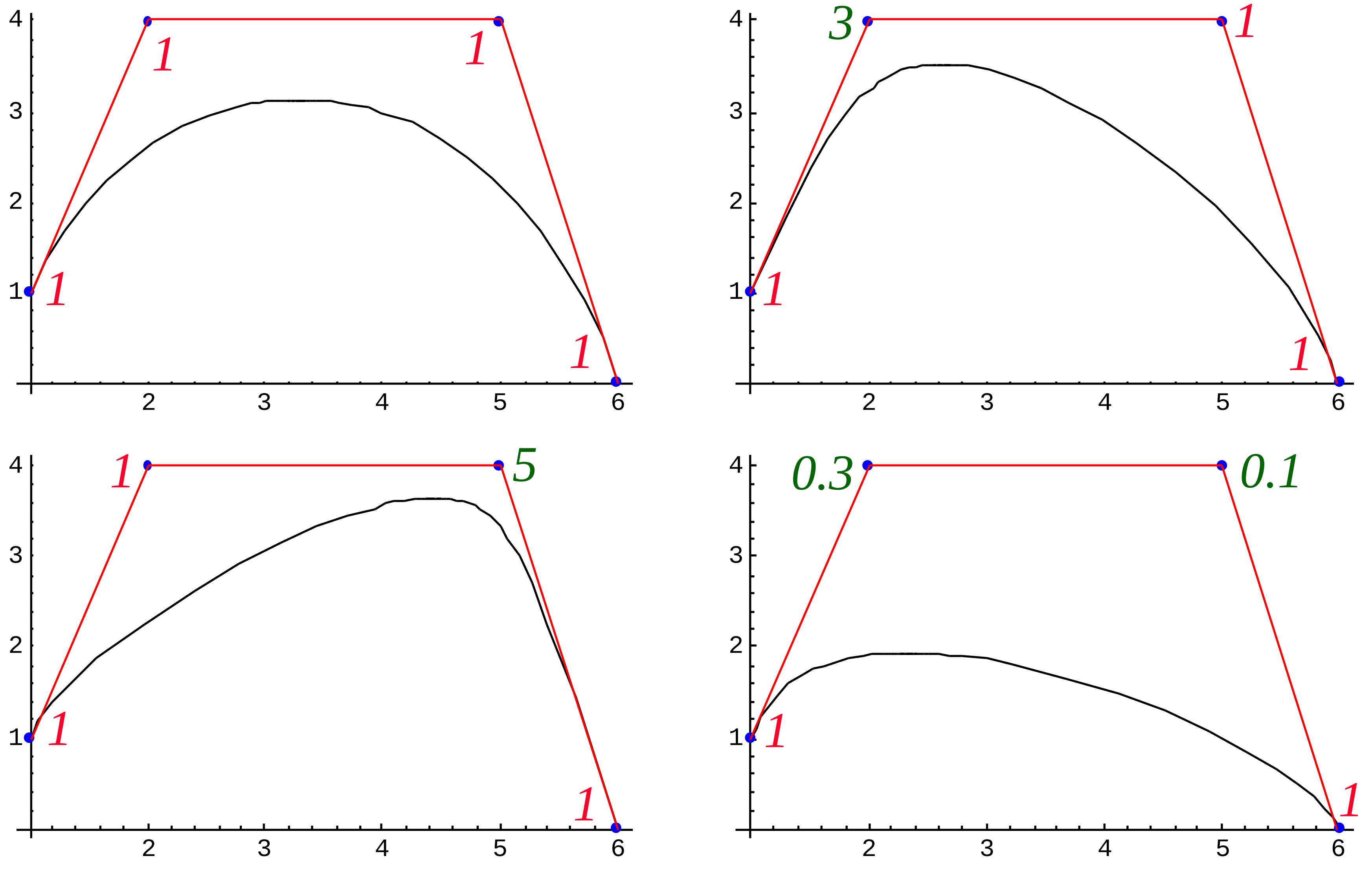
Farin, G.: *Curves and Surfaces for CAGD*, Academic Press, 3rd. Edition, 1993 (Chapters 14 and 15).
Hoschek, J. and Lasser, D.: *Fundamentals of CAGD*, A.K. Peters, 1993 (Chapter 4).
Anand, V.: *Computer Graphics and Geometric Modeling for Engineers*, John Wiley & Sons, 1993 (Chapter 10).

Rational Bézier curves

Changing the weights:

$w_i > 1$ -> the curve *approximates* to P_i

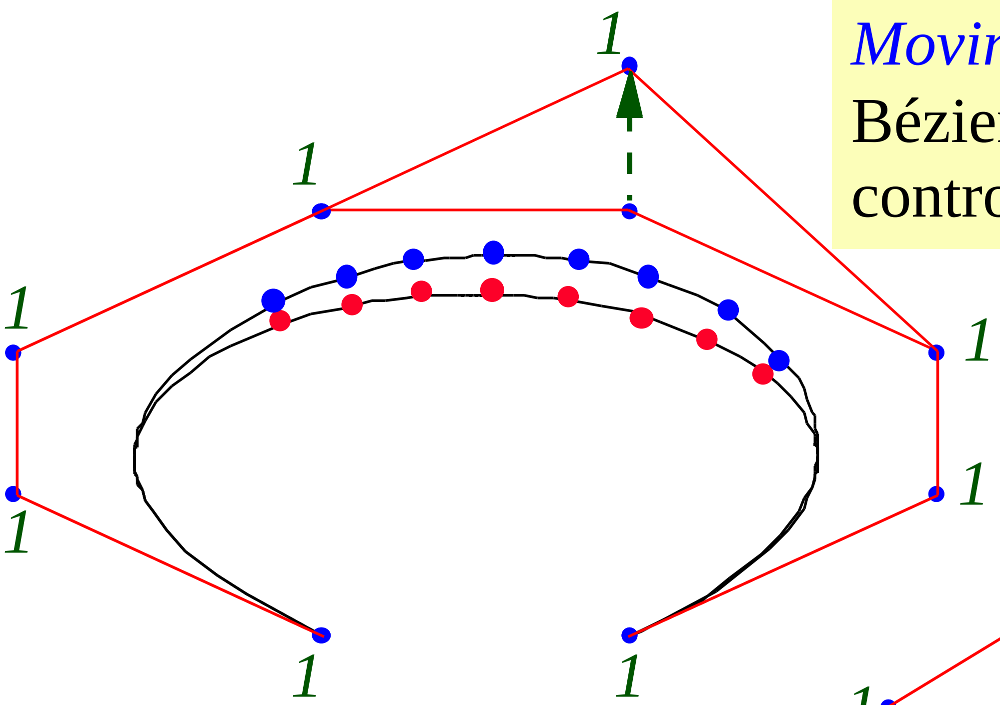
$w_i < 1$ -> the curve *moves away* from P_i



Rational Bézier curves

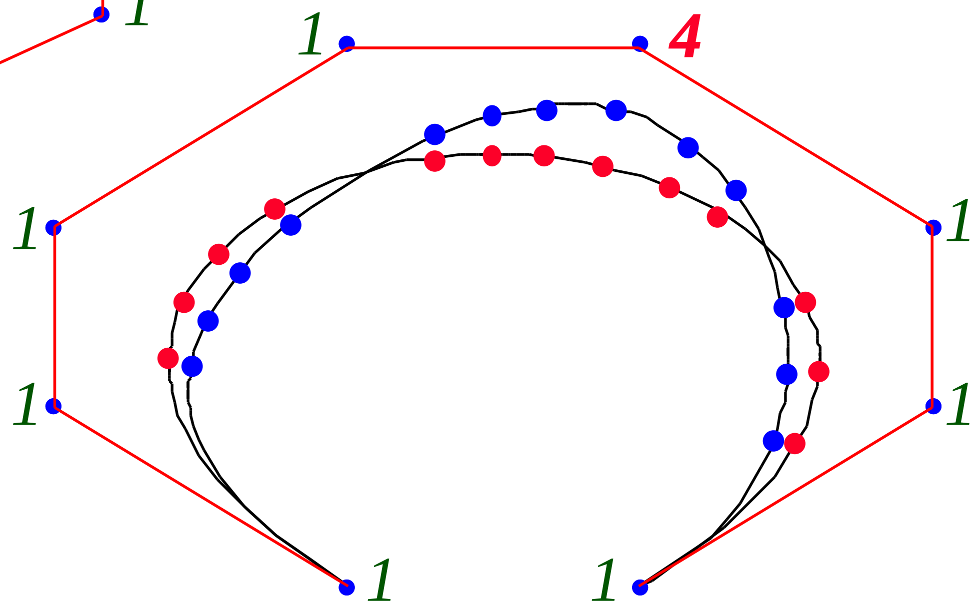
Influence of the *weights*:

The effect of changing a weight is different from that of moving a control point.



Moving a control point: a nonrational Bézier curve with a change in one control point.

● Original curve
● Final curve



Changing a weight: a rational Bézier curve with one weight changed.

Rational Bézier curves

Rational Bézier curves are useful to represent conics, which become an important tool in the aircraft industry.

Let $c(t)$ be a point on a conic. Then, there exist numbers w_0, w_1 and w_2 and two-dimensional points P_0, P_1 and P_2 such that:

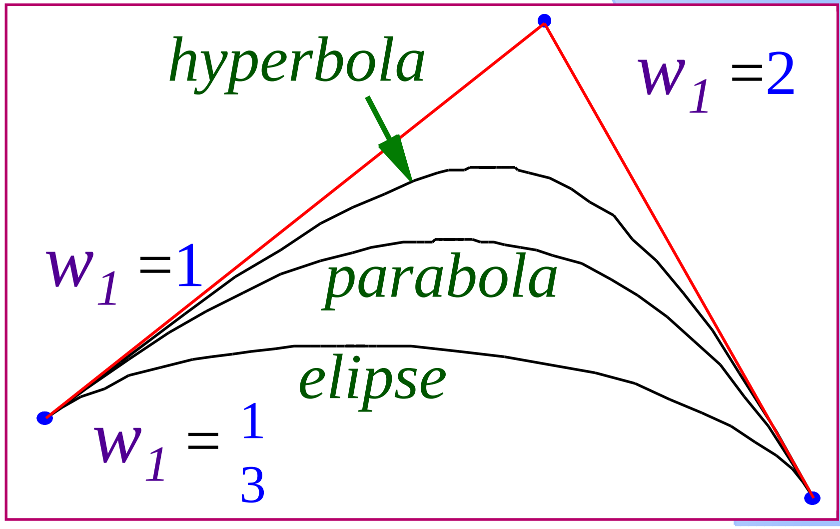
$$c(t) = \frac{w_0 P_0 B_0^2(t) + w_1 P_1 B_1^2(t) + w_2 P_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

If we take $w_0 = w_2 = 1$ and we define $s = \dfrac{w_1}{1+w_1}$:

$s = \dfrac{1}{2}$ gives a *parabolic* arc

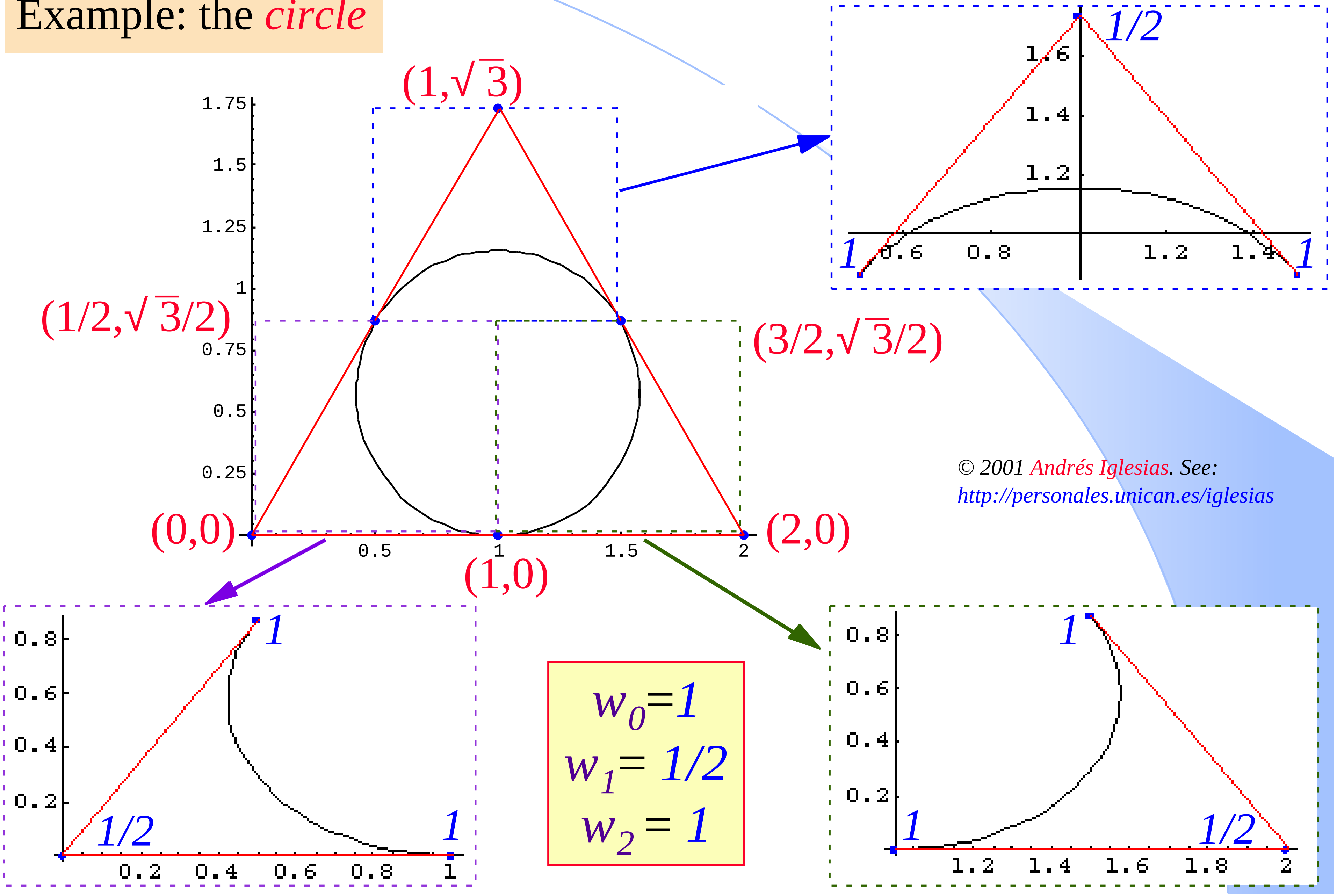
$s < \dfrac{1}{2}$ gives an *elliptic* arc

$s > \dfrac{1}{2}$ gives a *hyperbolic* arc



Rational Bézier curves

Example: the *circle*



$w_0=1$
$w_1=1/2$
$w_2=1$

Bézier surfaces

BEZIER SURFACES

Let $P=\{\{P_{00},P_{01},\ldots,P_{0n}\},$
$\{P_{10},P_{11},\ldots,P_{1n}\},$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$
$\{P_{m0},P_{m1},\ldots,P_{mn}\}\}$

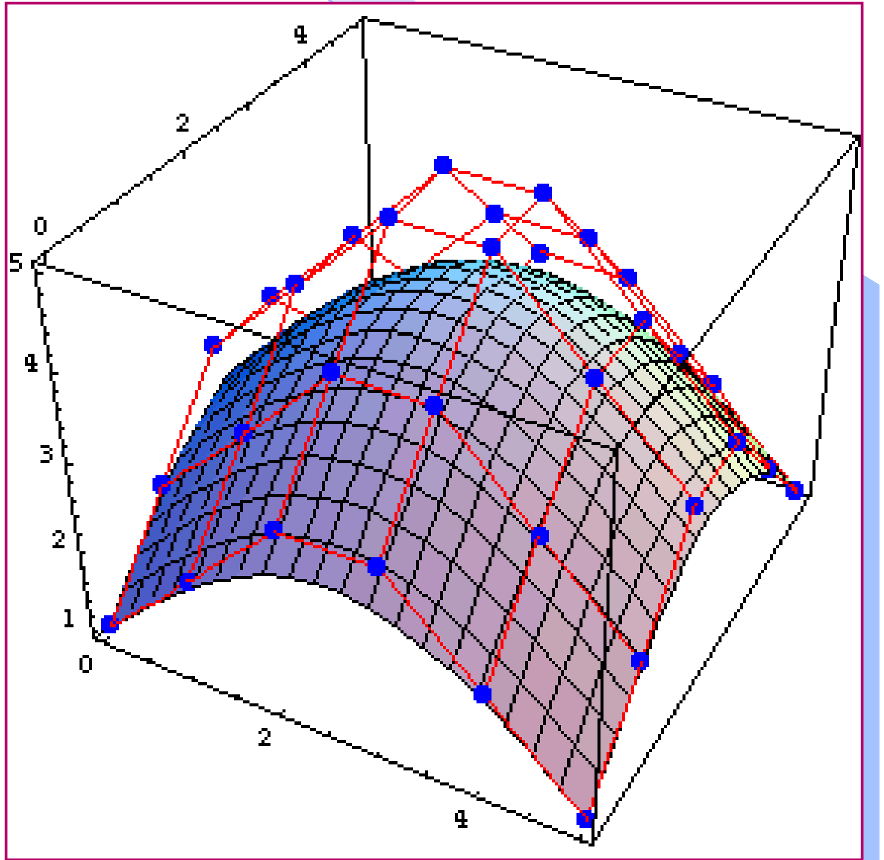
be a set of points $\mathbf{P}_{ij} \in \mathbb{R}^3$
$(i=0,1,\ldots,m \;;\; j=0,1,\ldots n)$

The Bézier surface associated with the set P is defined by:

$$\mathbf{S}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{ij}B_i^m(u)B_j^n(v)$$

where $B_i^m(u)$ and $B_j^n(v)$ represent the Bernstein polynomials of degrees m and n and in the variables u and v, respectively.

x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z
0	0	1	0	1	2	0	2	3	0	3	3	0	4	2	0	5	1
1	0	2	1	1	3	1	2	4	1	3	4	1	4	3	1	5	2
2	0	3	2	1	4	2	2	5	2	3	5	2	4	4	2	5	3
3	0	3	3	1	4	3	2	5	3	3	5	3	4	4	3	5	3
4	0	2	4	1	3	4	2	4	4	3	4	4	4	3	4	5	2
5	0	1	5	1	2	5	2	3	5	3	3	5	4	2	5	5	1



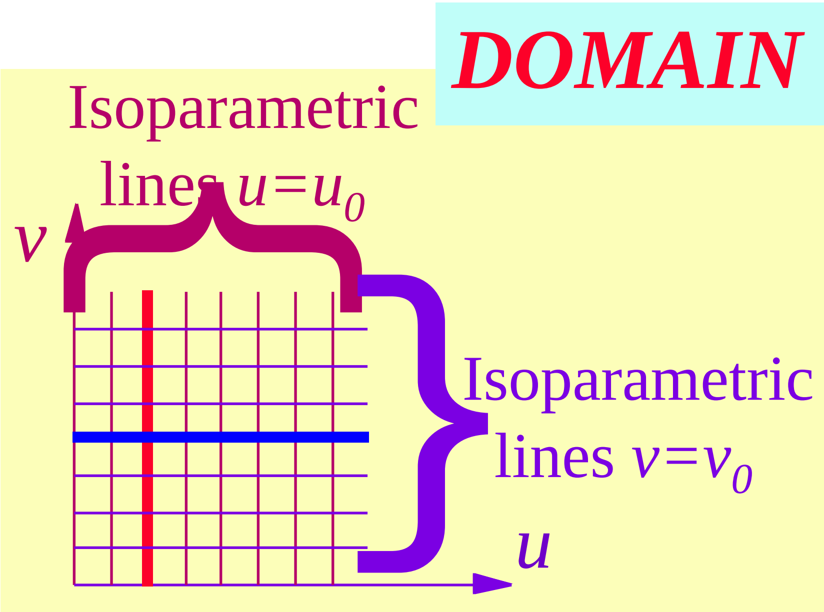
Bézier surfaces

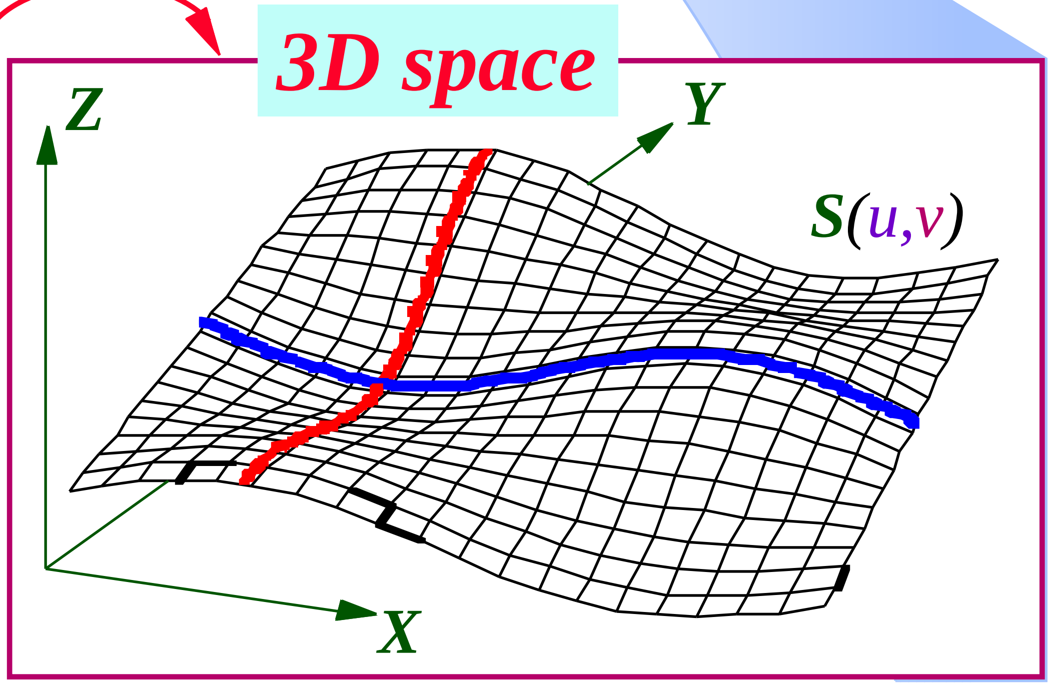
Note that along the *isoparametric lines* $u=u_0$ and $v=v_0$, the surface reduces to Bézier curves:

$$\mathbf{S}(u_0, v) = \sum_{j=0}^{n} b_j \; B_j^n(v) \qquad \mathbf{S}(u, v_0) = \sum_{i=0}^{m} c_i \; B_i^m(u)$$

with control points:

$$\boxed{b_j = \sum_{i=0}^{m} P_{ij} \; B_i^m(u_0)} \qquad \boxed{c_i = \sum_{j=0}^{n} P_{ij} \; B_j^n(v_0)}$$

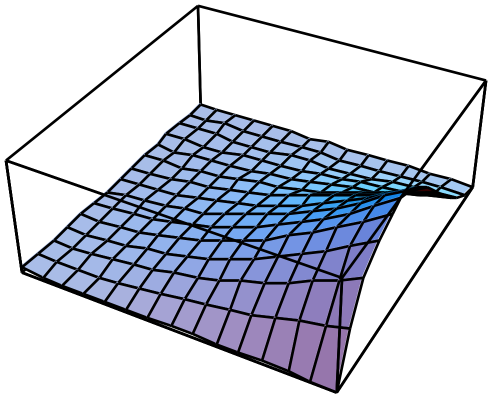


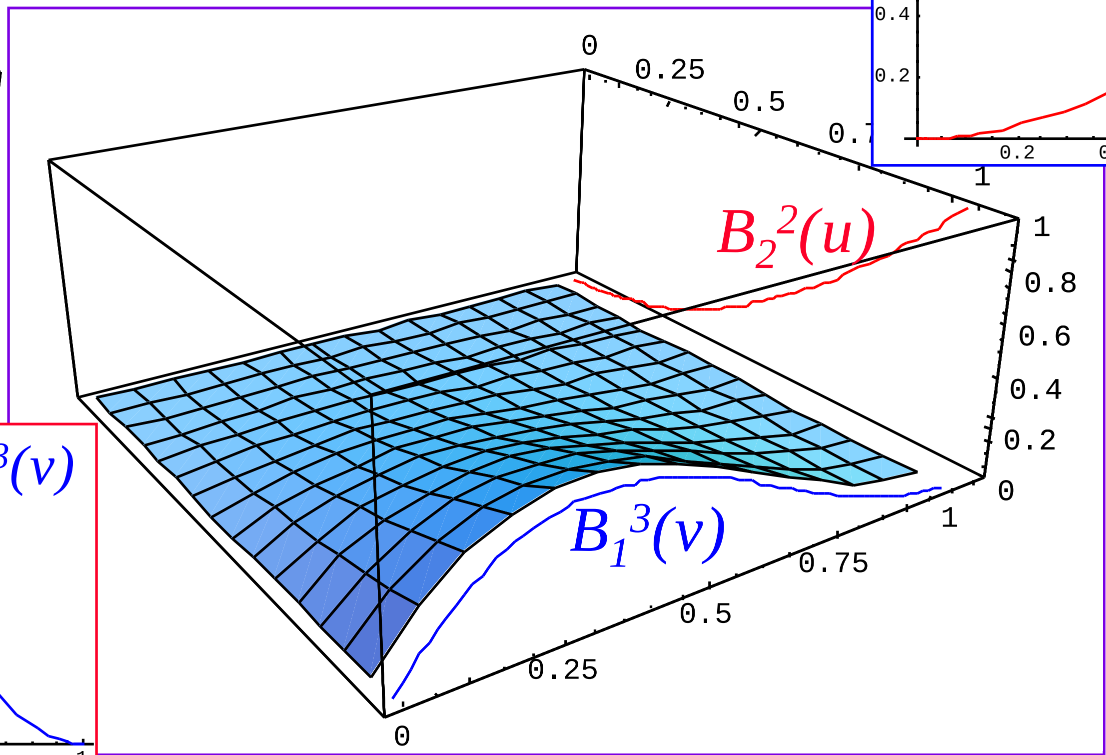


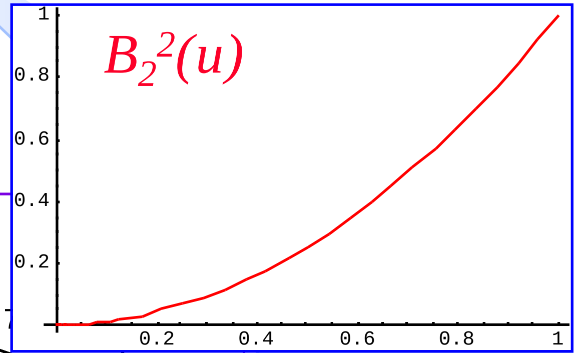
Bézier surfaces

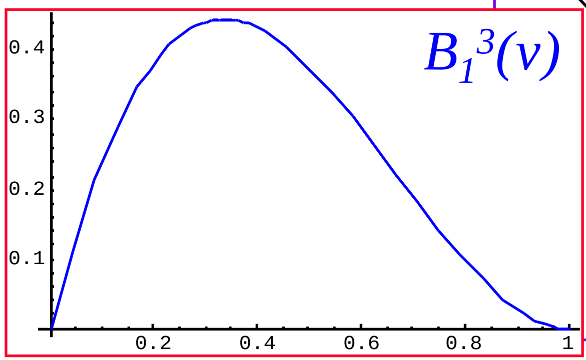
From the equation: $\mathbf{S}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{ij}B_i^m(u)B_j^n(v)$ it is clear that each term is obtained from a control point and the product of two univariate Bernstein polynomials. Each product makes up a basis function of the surface. For instance:

Function $B_2^2(u)$. $B_1^3(v)$:







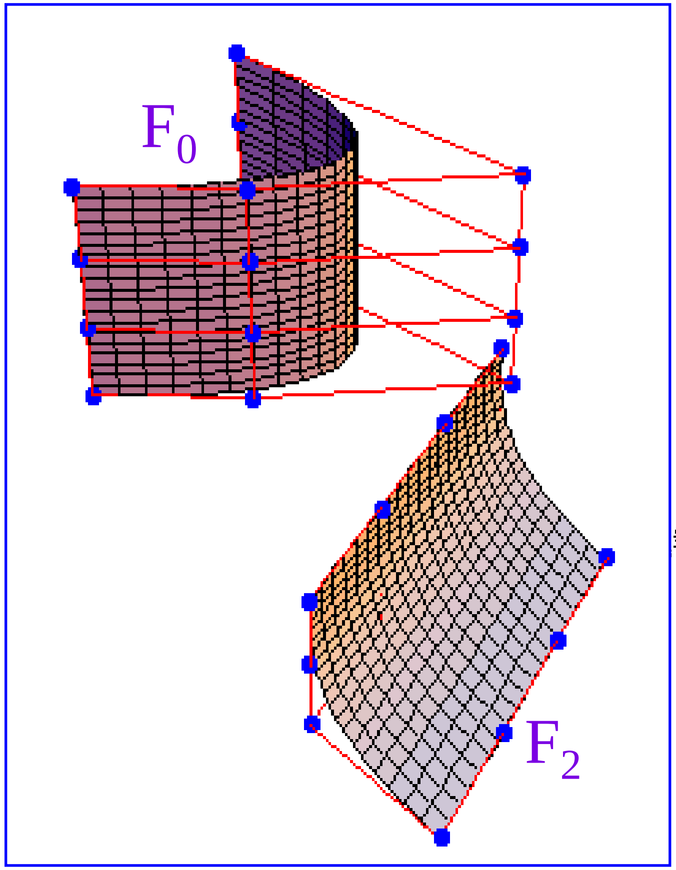


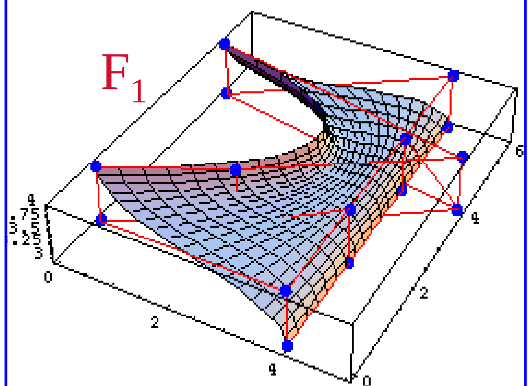
Bézier surfaces

BLENDING SURFACES

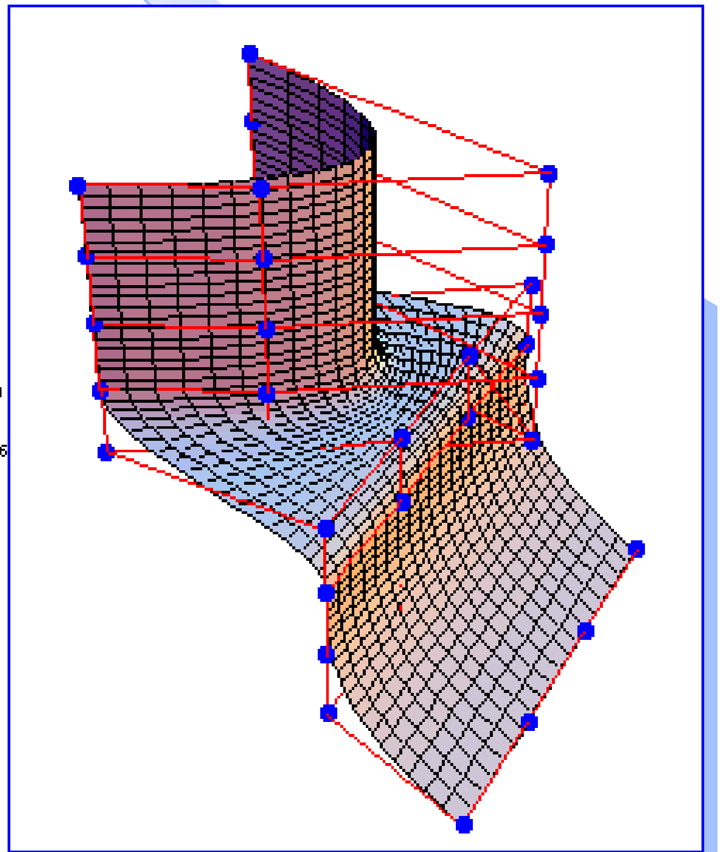
If a single surface does not approximate enough a given set of points, we may use several patches joined together.

Two Bézier patches F_0 and F_2 are connected with C^1-continuity by using a Bézier F_1 patch.





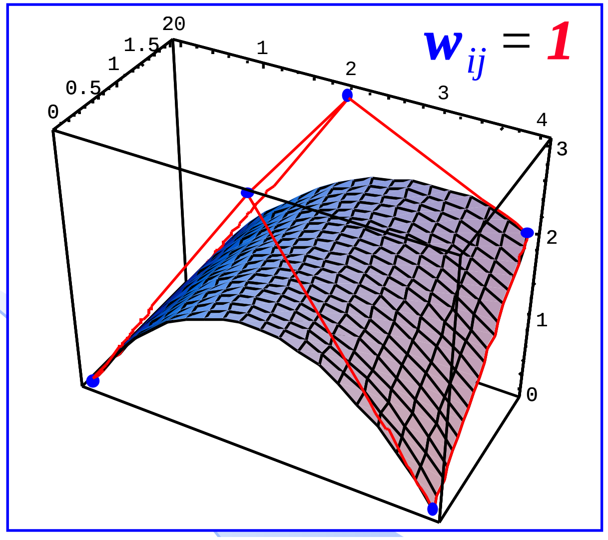
BLENDING



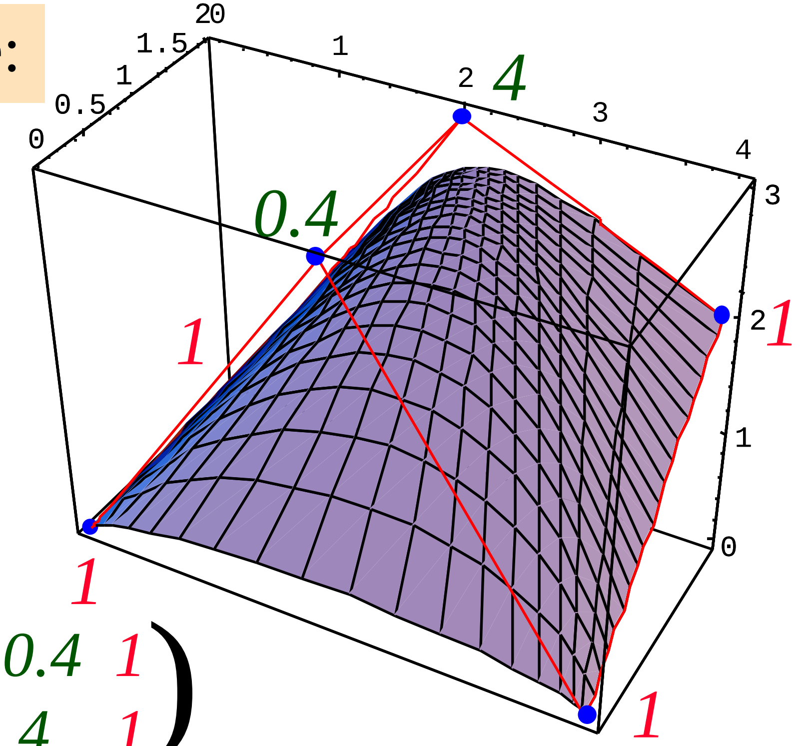
Rational Bézier surfaces

If we introduce weights w_{ij} to a nonrational Bézier surface, we obtain a rational Bézier surface:

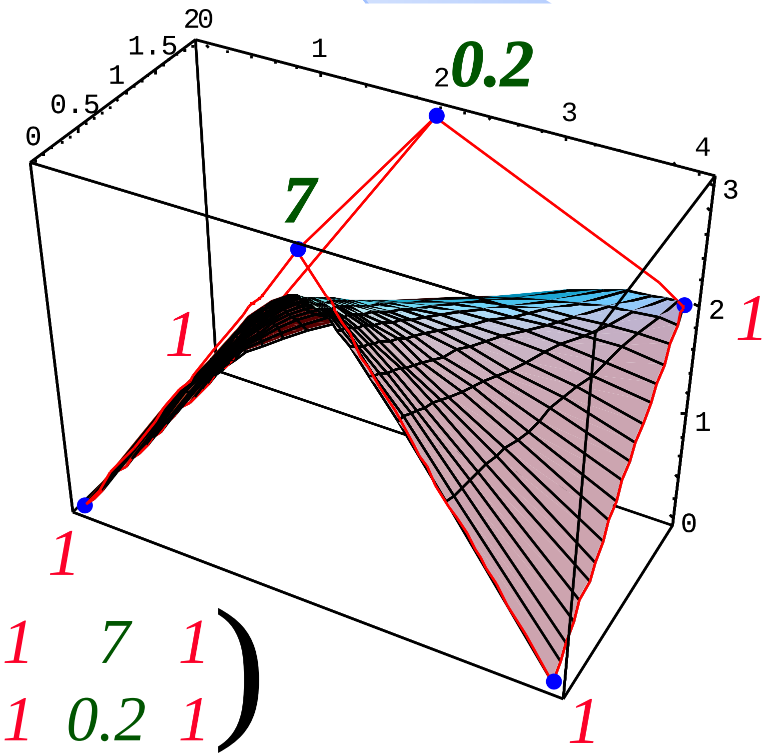
$$\mathbf{S}(u,v)=\frac{\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{ij}\,w_{ij}\,B_i^m(u)B_j^n(v)}{\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}w_{ij}\,B_i^m(u)B_j^n(v)}$$



Example:



$$\begin{pmatrix} 1 & 0.4 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 7 & 1 \\ 1 & 0.2 & 1 \end{pmatrix}$$