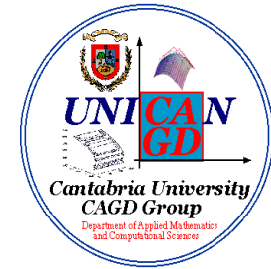




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**COMPUTER-AIDED GEOMETRIC DESIGN
AND COMPUTER GRAPHICS:
B-SPLINES AND NURBS CURVES
AND SURFACES**

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B-splines Curves

Knot vector: a list $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{m-1} \leq t_m$ of $m+1$ **nondecreasing** numbers, such that **the same value** should not appear more than k times, (k = **order** of the B-spline).

We define the i -th **B-spline function** $N_{ik}(t)$ of **order** k ($= k-1$ degree) as:

$$N_{i1}(t) = \begin{cases} 1 & \text{if } t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad k=1$$

$$N_{ik}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \quad k > 1$$

1. $N_{ik}(t) > 0$ for $t_i < t < t_{i+k}$
2. $N_{ik}(t) = 0$ for $t_0 \leq t \leq t_i, t_{i+k} \leq t \leq t_{n+k}$
3. $\sum_{i=0}^n N_{ik}(t) = 1 \quad t \in [t_{k-1}, t_{n+1}]$ **Normalizing property**

B-splines Curves

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Given a set of $n+1$ control points (called *de Boor points*) \mathbf{d}_i ($i=0, \dots, n$) and a knot vector $T=[t_0, t_1, \dots, t_{m-1}, t_m]$ we define a **B-spline $X(t)$ of order k** as:

$$\mathbf{X}(t) = \sum_{i=0}^n \mathbf{d}_i N_{ik}(t)$$

where $N_{ik}(t)$ describes the blending B-spline function of degree $k-1$ associated with the knot vector T .

PROPERTIES:

- (a) The degree of the polynomial does not exceed $k-1$.
- (b) The first $k-2$ derivatives are continuous.

Knot vector

$$m+1 = (n+1) + k$$

of knots # of control points order of curve

$$m = n + k$$

Knot vectors can be classified as:

Periodic / Uniform

$$t_i - t_{i-1} = C$$

B-splines functions are all translates of each other: $n=3, k=3$ [0 1 2 3 4 5 6]

NonPeriodic

k repeated values at the ends ($k = \text{order}$)

$$n=3, k=3 \quad [0 \ 0 \ 0 \ 1 \ 2 \ 2 \ 2]$$

NonUniform

NURBS

B-splines Curves

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Knot vector

Periodic / Uniform

- Influence of each basis function is limited to k intervals

- Parameter range: $(k-1) \quad t \quad (n+1)$

EXAMPLE $[0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6]$

NonPeriodic

- No loss of parameter range: the curve interpolates the first and the last control points

- Parameter range: $0 \quad t \quad n-k+2$

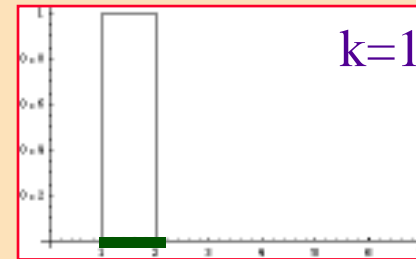
EXAMPLE $[0 \ 0 \ 0 \ 1 \ 2 \ 2 \ 2]$

NonUniform (NURBS)

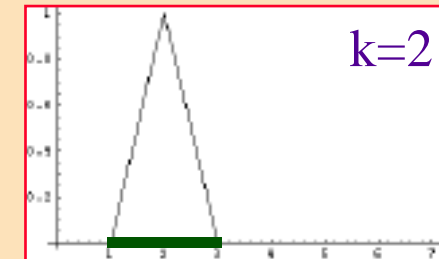
EXAMPLES $[0 \ 1 \ 2 \ 3 \ 3 \ 3 \ 4]$
 $[0 \ 0 \ 0 \ 1 \ 2 \ 5 \ 5]$

EXAMPLE

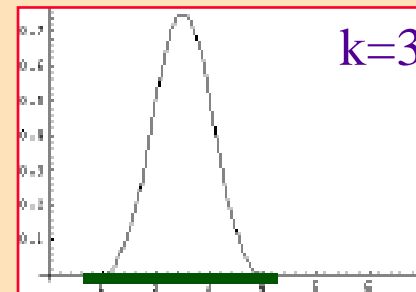
B-splines $B_{2k}(t)$



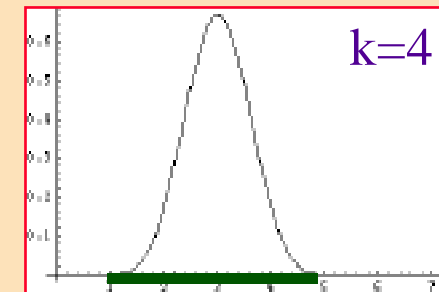
1



2



3



4

Bézier representation

It is a simple case of B-splines, when:

- # of control points = order B-spline
- A nonperiodic knot vector is considered

B-splines Curves

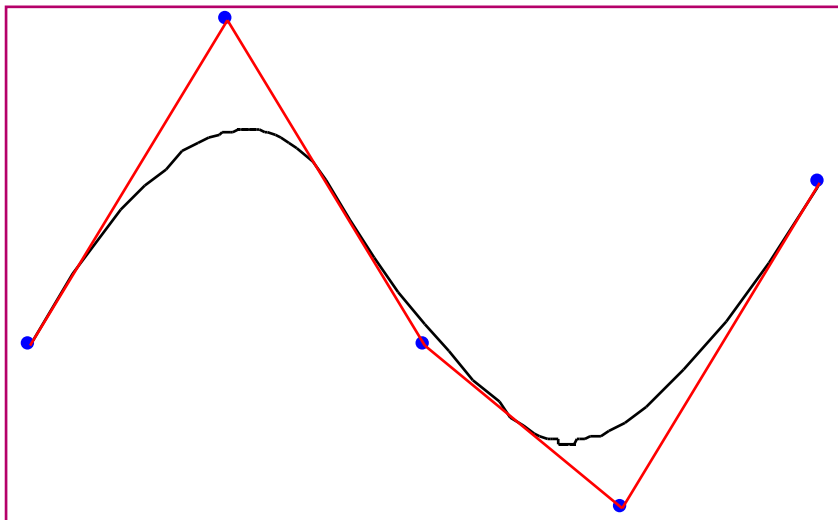
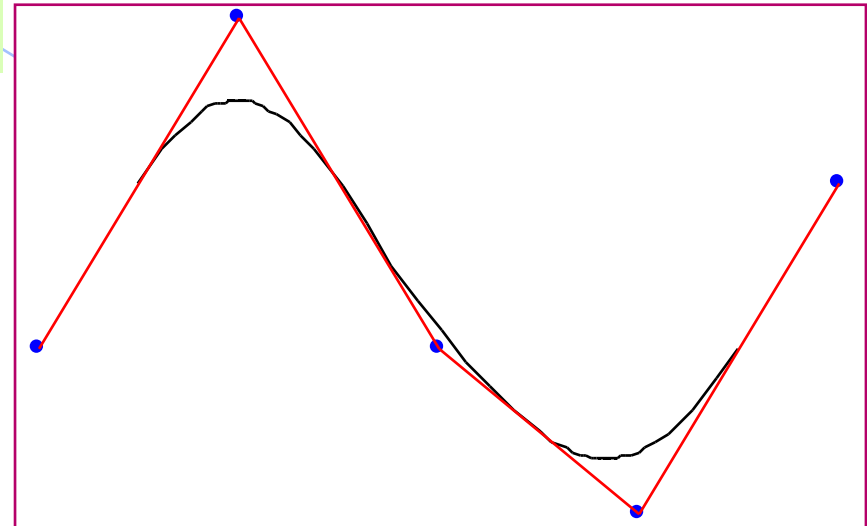
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Example: Five control points $n=4$
 Quadratic B-spline: $k=3$

$P=((1,2),(2,4),(3,2),(4,1),(5,3))$

Periodic / Uniform $T=[0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7]$

Parameter range: $(2,5)$
 Curve *does not interpolate* the end points.



NonPeriodic $T=[0 \ 0 \ 0 \ 1 \ 2 \ 3 \ 3 \ 3]$

Parameter range: $(0,3)$
 Curve *interpolates* the end points.

For the case $n=k$, we recover the *Bézier curves*.

NonUniform

$T=[0 \ 1 \ 2 \ 3 \ 4 \ 4 \ 4 \ 5]$

$T=[0 \ 2 \ 2 \ 5 \ 5 \ 5 \ 26]$

$T=[0 \ 0 \ 0 \ 1 \ 2 \ 5 \ 6 \ 6]$

$T=[0.3 \ 2 \ 2 \ 8 \ 8 \ 9.6 \ 9.6]$

Rational B-splines Curves

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Introducing weights w_{ij} we obtain a rational B-spline curve given by:

$$R(t) = \frac{\sum_{i=0}^n P_i w_i N_{ik}(t)}{\sum_{i=0}^n w_i N_{ik}(t)}$$

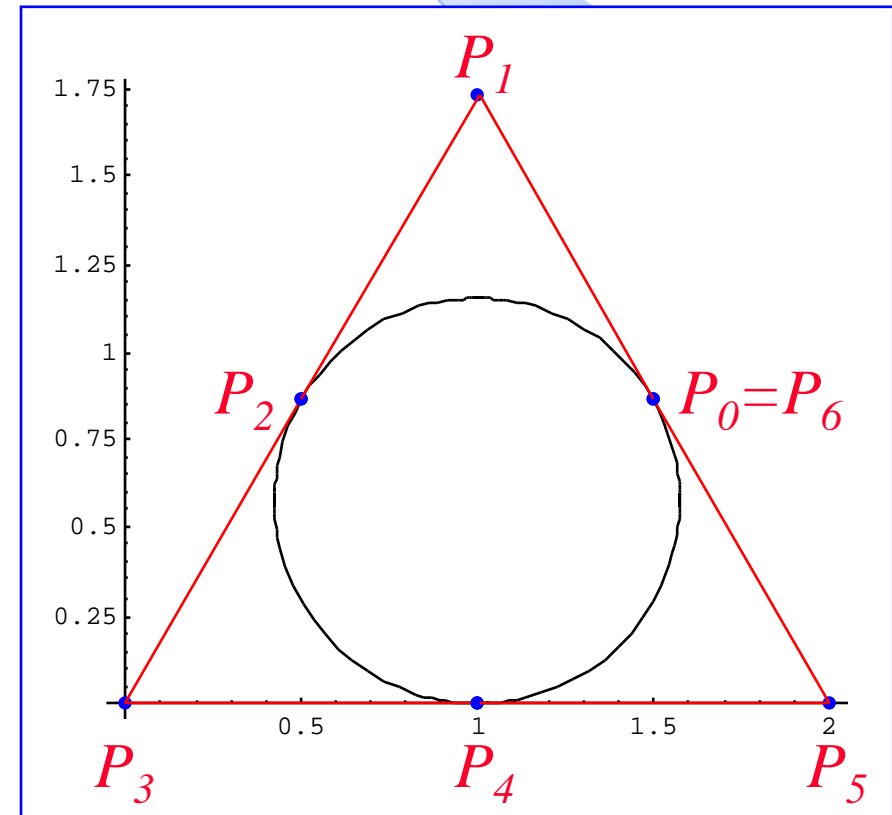
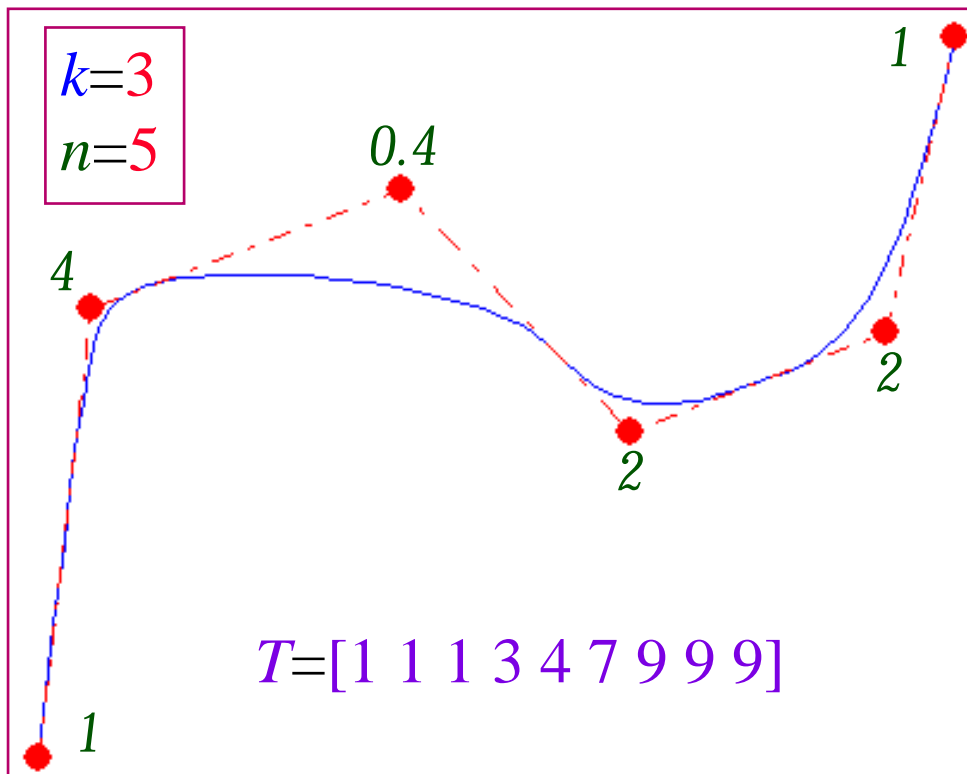
Example: A full circle can be obtained by using seven control points:

$$\{P_0, P_1, P_2, P_3, P_4, P_5, P_6\}$$

the knot vector:

$$T=[0 \ 0 \ 0 \ 1/3 \ 1/3 \ 2/3 \ 2/3 \ 1 \ 1 \ 1]$$

and weights: $w=\{1, 1/2, 1, 1/2, 1, 1/2, 1\}$

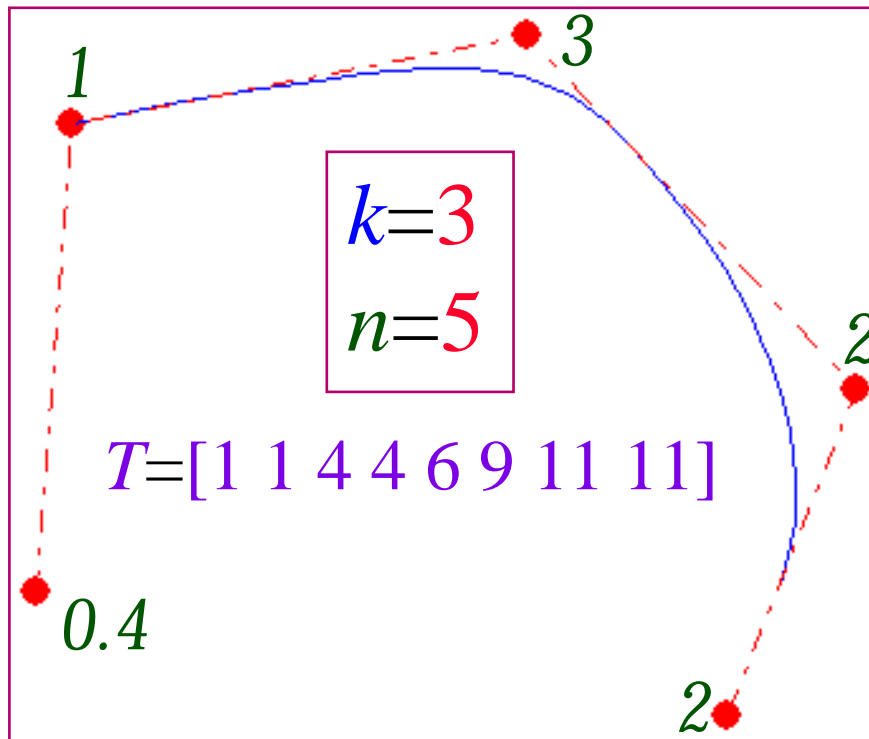


Non-Uniform Rational B-splines (NURBS)

Today, NURBS become the most important geometric entity in design.

Incorporated in the most popular CAD/CAM and Computer Graphics systems

A NURBS curve with no interior knots is a rational Bézier curve. So, NURBS curves contain nonrational B-splines and rational and nonrational Bézier curves as special cases.



Unfortunately, it is no easy to understand how they work

*Non
Uniform
Rational
B-
Splines*

@

*Nobody
Understands
Rational
B-
Splines*

Piegl, L. and Tiller, W: *The NURBS Book*, 2nd. Edition, Springer Verlag Berlin, 1997.

B-splines Surfaces

A **B-spline surface** S of order k in the u direction and order l in the v direction is a bivariate vector-valued piecewise function of the form:

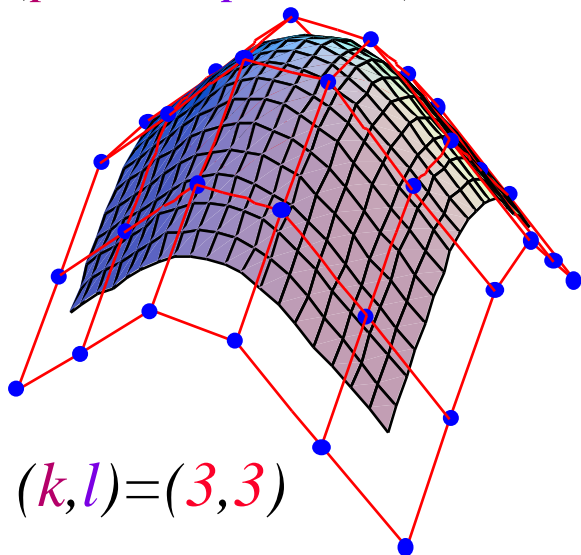
$$S(u, v) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} P_{ij} N_{ik}(u) N_{jl}(v)$$

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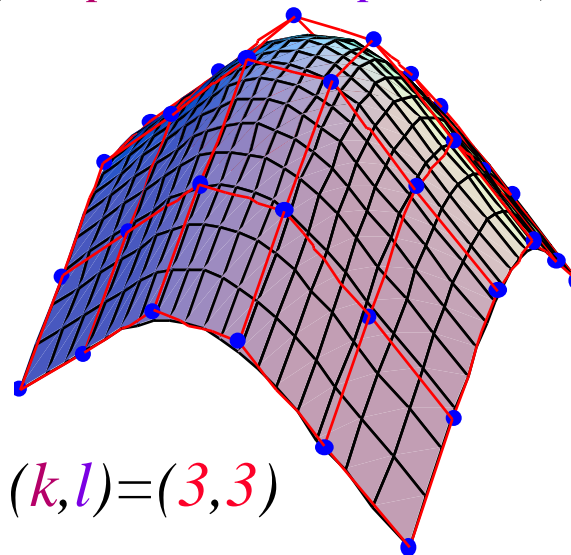
where the P_{ij} form a bidirectional net and the $N_{ik}(u)$ and $N_{jl}(v)$ are the B-spline functions defined on the knot vectors:

$$U = [u_0, u_1, \dots, u_{p-1}, u_p] \quad \text{and} \quad V = [v_0, v_1, \dots, v_{q-1}, v_q] \quad (p = m+k, q = l+n)$$

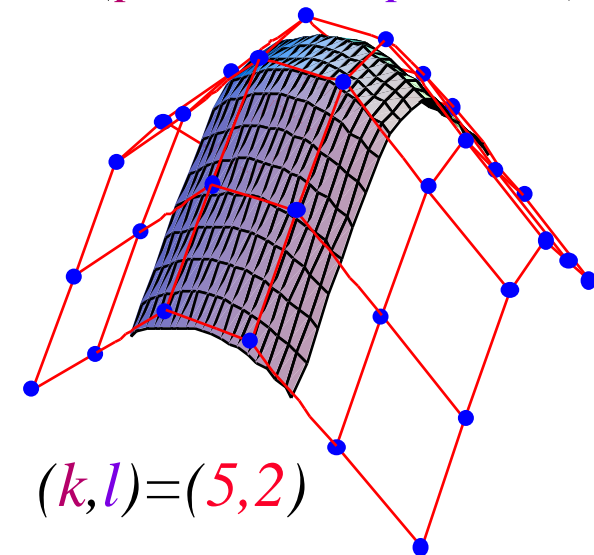
(periodic, periodic)



(nonperiodic, nonperiodic)



(periodic, nonperiodic)



Properties of the B-splines Surfaces

1. If $U = [0, \dots, 0, 1, \dots, 1]$ and $V = [0, \dots, 0, 1, \dots, 1]$ then the surface S interpolates the four corner points:

$$S(0,0) = P_{00}, \quad S(1,0) = P_{m0}, \quad S(0,1) = P_{0n}, \quad S(1,1) = P_{mn}$$

2. If $m=k$, $n=l$, $U = [0, \dots, 0, 1, \dots, 1]$ and $V = [0, \dots, 0, 1, \dots, 1]$ then S is a Bézier surface.

3. *Affine invariance*: an affine transformation is applied to the surface by applying it to the control points.

4. *Strong convex hull property*: if (u,v) belongs to $[u_r, u_{r+1}] \times [v_s, v_{s+1}]$, S is in the convex hull of the control points P_{ij} ($r-k \leq i < r$, $s-l \leq j < s$)

5. *Local modification scheme*: if P_{ij} is moved, it affects the surface only in the rectangle $[u_i, u_{i+k+1}] \times [v_j, v_{j+l+1}]$.

6. Isoparametric curves for S behave analogously to that the Bézier surfaces.

Rational B-splines Surfaces

A rational B-spline surface S of order k in the u direction and order l in the v direction is defined by:

$$S(u, v) = \frac{\sum_{j=0}^m \sum_{i=0}^n P_{ij} w_{ij} N_{ik}(u) N_{jl}(v)}{\sum_{j=0}^m \sum_{i=0}^n w_{ij} N_{ik}(u) N_{jl}(v)}$$

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where the P_{ij} , $N_{ik}(u)$ and $N_{jl}(v)$ are given as for the nonrational case, and w_{ij} are the weights, which will be assumed $w_{ij} > 0$, for all i, j .

Once again, if all $w_{ij} = 1$, we recover the nonrational B-spline surfaces.

NURBS surfaces are included here as a particular case. However, some authors *identify rational B-spline surfaces to NURBS surfaces*. From this point of view, NURBS surfaces are a key topic in computer graphics, being incorporated in most of the computer design systems.

Piegl, L. and Tiller, W: *The NURBS Book*, 2nd. Edition, Springer Verlag Berlin, 1997.

Applications to industry

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